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On the Construction of Survivorship Assurance Tables.

By PETER GRAY, ESQ., F.R.A.S.

[Read before the Institute, 26th April, 1852, and ordered by the Council to be printed.]

IN a Postscript to the introductory matter of the *Assurance and Annuity Tables*, compiled and published by Messrs. Smith, Orchard, and myself in April, 1851 (and which Postscript was written by me), it is stated that I had “devised a new table, which, while answering all, and more than all, the purposes of Tables V. and VI. [which show the single and annual premiums for a survivorship assurance of £1 for every combination of two ages], by means of a *single* value for each pair of ages, and thus occupying little more space than one of them, would also be very much less laborious in its construction than the former of those tables. The new tables would, in fact, consist of series of columns supplementary to Mr. Jones’s *Commutation Tables for Two Lives*, and possessing all the properties, *mutatis mutandis*, with respect to the formation of temporary and deferred, as well as present, whole life benefits and payments, that give to tables of the form of those mentioned their great value.” The form of the table to which reference is here made suggested itself to me while occupied in superintending the printing of the work which contained the foregoing announcement. Having, however, had just then, as I thought, quite enough to do with survivorship assurances for a time, I contented myself with making the announcement cited, and communicating my ideas in a general way to my coadjutors, Messrs. Smith and Orchard.

My attention has recently been recalled to the subject—first, by a suggestion in the January Number of the *Assurance Magazine* that I ought to make my method public, and, secondly, by the reading of a paper by Mr. David Chisholm, at the meeting of the Institute of Actuaries on the 23rd of February last, in which a method of constructing the same tables that I had in view is very fully described and exemplified. Mr. Chisholm’s paper appeared to me to be an exceedingly able one; and being occupied with a subject to which I had devoted great attention, I listened to the reading of it with much interest. In particular, I was gratified by the recognition, on the part of a gentleman of so much experience, of the value of, and the desirableness of possessing, tables of the kind I was the first to suggest.

Mr. Chisholm's method of construction differs from mine. First, his method is essentially logarithmic, while mine admits of being applied both in numbers and in logarithms; and of the two modes I think the former is the preferable. Again, Mr. C.'s method is not *continuous*—that is, each tabular value is formed independently of other tabular values; and as a consequence, to ensure accuracy, either the whole work must be subjected to a rigid scrutiny, or the entire computation must be performed in duplicate. My method, on the other hand, in both its applications, is continuous. Each succeeding value is deduced, by the simplest conceivable operation, from a preceding value, so that verification at a few points suffices to verify the whole. It seems to me, therefore, that it may not be unacceptable if I proceed now to explain and exemplify the method I had in view. Those who may be disposed to undertake the improving exercise of forming the tables in question, will at all events have thus a choice of methods presented for their adoption.

It is known that

$$\frac{1}{2}(p_{x \cdot n-1} - p_{x \cdot n})(p_{y \cdot n-1} + p_{y \cdot n})v^n,$$

which may be also written thus,

$$\frac{\frac{1}{2}(l_{x+n-1} - l_{x+n})(l_{y+n-1} + l_{y+n})v^n}{l_{x \cdot y}}$$

denotes the present value of £1 to be received at the end of n years, provided (x) shall have died in the n th year, and (y) have been alive at the instant of (x) 's death. Now to adapt this expression to Mr. Jones's tables, we must introduce in both numerator and denominator such a power of v as will make the latter equal to $D_{x \cdot y}$. This will be either v^x or v^y , according as x or y is the greater. We shall thus have for the value of the contingency under consideration in the two cases—

$$\frac{(x > y)}{\frac{1}{2}(l_{x+n-1} - l_{x+n})(l_{y+n-1} + l_{y+n})v^{x+n}}; \frac{(x < y)}{\frac{1}{2}(l_{x+n-1} - l_{x+n})(l_{y+n-1} + l_{y+n})v^{y+n}}.$$

The denominators of these expressions, as they now stand, being already formed (in Jones's tables), it is obvious that we have now to occupy ourselves with the formation of the numerators only, for the requisite number of cases of variation of x , y , and n .

Looking, then, at the numerators, involving three variables, x , y , and n , it might at first sight seem to be requisite to form

them for every value of each of these quantities. But, in the first place, since of the two quantities x and y , neither takes any values different from those taken by the other, and since in the first expression each value of x is combined only with those values of y that do not exceed it, and in the second each value of y is combined only with those values of x that do not exceed it, it follows that the number of formations is thus at once reduced one half. And, in the second place, since, *owing to the peculiar structure of the Commutation Tables*, in both expressions the same effect is produced by any variation given to n as by an *equal* variation given to both x and y , it further follows, that it will suffice to compute the two expressions for all the values of x and y , and for a single value of n .

It is most convenient to make $n=1$, in which case the numerators of the two expressions (which will then denote the value of the contingencies in respect of the first year) will respectively become

$$\begin{array}{cc} (x \nabla y) & (x \angle y) \\ \frac{1}{2} (l_x - l_{x+1}) (l_y + l_{y+1}) v^{x+1}, & \text{and } \frac{1}{2} (l_x - l_{x+1}) (l_y + l_{y+1}) v^{y+1}. \end{array}$$

And, in accordance with what has just been said, the values of these expressions that will have to be computed, are (supposing the Carlisle Table to be used), of the first, those in which x receives every value from 104 to 0, and y every value from x to 0; and, of the second, those in which y receives every value from 104 to 0, and x every value from y to 0. I proceed now to inquire how these values may be most advantageously formed.

The expressions deduced above will be simplified, if for $\frac{1}{2} (l_x - l_{x+1})$ and $l_y + l_{y+1}$, we write d_x and s_y respectively. It might seem preferable to combine the constant factor $\frac{1}{2}$ with the quantity $l_y + l_{y+1}$, inasmuch as d_x and s_y would then denote, the former the number dying between the ages x and $x+1$, and the latter the number living at age $y + \frac{1}{2}$. But the other arrangement will be found to be the more advantageous, since, the values of $l_y + l_{y+1}$ being generally very much greater than those of $l_x - l_{x+1}$, we are less inconvenienced, in the case of this latter function, by the additional figure which arises from the division of an odd number by 2. The expressions with which we have to deal will therefore now be denoted as follows:—

$$\begin{array}{cc} (x \nabla y) & (x \angle y) \\ d_x s_y v^{x+1}, & \text{and } d_x s_y v^{y+1}. \end{array}$$

The following table, formed from the Carlisle Table of Mortality by a process too obvious to need explanation, contains all the terms of the series s_y and d_x , with their differences, for which we shall have occasion in the subsequent examples.

$x \text{ \& } y$	s_y	Δs_y	d_x	Δd_x
104	1	3	·5	·5
103	4	4	1·0	·0
102	8	4	1·0	·0
101	12	4	1·0	·0
100	16	4	1·0	·0
99	20	5	1·0	·5
98	25	7	1·5	·5
97	32	9	2·0	·5
96	41	12	2·5	1·0
95	53	17	3·5	1·5
94	70	24	5·0	2·0
93	94	35	7·0	3·5
92	129	51	10·5	4·5
91	180	67	15·0	3·5
90	247	76	18·5	1·0
89	323	90	19·5	6·0
88	413	115	25·5	6·5
87	528	135	32·0	3·5
86	663	149	35·5	3·5
85	812	162	39·0	3·0
84	974	178	42·0	5·0
83	1152	196	47·0	4·0
82	1348		51·0	

At this point the two methods which I propose to develop part company, and must henceforth be treated separately. I therefore commence with—

I.—*The Construction in Numbers.*

The process is to be continuous. To see, therefore, how succeeding values can be most readily deduced from each other, taking first the case $x > y$, write down in order a few of the leading terms of the series belonging to adjoining differences of age, thus :—

$x - y = 0$	$x - y = 1$	$x - y = 2$	$x - y = 3$
$d_{104}s_{104}v^{105}$	$d_{104}s_{103}v^{105}$	$d_{104}s_{102}v^{105}$	$d_{104}s_{101}v^{105}$
$d_{103}s_{103}v^{104}$	$d_{103}s_{102}v^{104}$	$d_{103}s_{101}v^{104}$	$d_{103}s_{100}v^{104}$
$d_{102}s_{102}v^{103}$	$d_{102}s_{101}v^{103}$	$d_{102}s_{100}v^{103}$	$d_{102}s_{99}v^{103}$

and so on. Now what we have to ascertain is, how we can most easily pass from any one term to an adjoining term. Examine, then, the succession of values in the several *columns*. We see that in passing from term to term all the three factors vary; the

addends consequently would obviously be characterized by a degree of complexity that would quite unfit them for our purpose. But examine the succession in the several *rows*. Here we see that only one factor varies in passing from term to term; and that the terms consequently are nothing else than a series of multiples by this varying factor of the constant product of the two remaining factors. Farther, the differences of these terms will also be a series of multiples of the same constant product; so that, if this last series be formed, we shall be able, by continuous addition of its terms to the initial value, namely, that corresponding to $x-y=0$, to form, with great facility, the terms belonging to each row.

To make this more plain, denote for a moment $d_{104}v^{105}$, $d_{103}v^{104}$, $d_{102}v^{103}$, &c., by P, Q, R, &c., respectively; then will the above-written terms take the following form:—

$x-y=0$	$x-y=1$	$x-y=2$	$x-y=3$
$P_{s_{104}}$	$P_{s_{103}}$	$P_{s_{102}}$	$P_{s_{101}}$
$Q_{s_{103}}$	$Q_{s_{102}}$	$Q_{s_{101}}$	$Q_{s_{100}}$
$R_{s_{102}}$	$R_{s_{101}}$	$R_{s_{100}}$	$R_{s_{99}}$

and so on. And their differences, taken from the several rows, will be

$P(s_{103}-s_{104}),$	$P(s_{102}-s_{103}),$	$P(s_{101}-s_{102}),$
$Q(s_{102}-s_{103}),$	$Q(s_{101}-s_{102}),$	$Q(s_{100}-s_{101}),$
$R(s_{101}-s_{102}),$	$R(s_{100}-s_{101}),$	$R(s_{99}-s_{100});$

or,

$P\Delta s_{104},$	$P\Delta s_{103},$	$P\Delta s_{102},$
$Q\Delta s_{103},$	$Q\Delta s_{102},$	$Q\Delta s_{101},$
$R\Delta s_{102},$	$R\Delta s_{101},$	$R\Delta s_{100}.$

It thus appears that the values occupying the first row in the foregoing scheme will be formed by the continuous addition to the initial term, $P_{s_{104}}$, of the products of the quantity P (or $d_{104}v^{105}$), by the successive terms of the series Δs_y , which is one of those of which a portion is tabulated on page 110. In like manner, the values occupying the second row will be formed by the continuous addition to *its* initial term, $Q_{s_{103}}$, of the products of the quantity Q (or $d_{103}v^{104}$), by the successive terms of the same series Δs_y ; and so also for the third and following rows.

A like property holds with respect to the function belonging to the case $x < y$, as may be very briefly shown.

$x \geq y$

$x - y$	s_y	$x = 104$	$x = 103$	$x = 102$	$x = 101$	$x = 100$
0	1	·0224420	·1849220	·3809394	·5885513	·8082771
	3	673260	·1849220	·1904697	·1961838	·2020693
1	4	·0897680	·3698440	·5714091	·7847351	1·0103464
	4	897680	·1849220	·1904697	·1961838	·2525866
2	8	·1795360	·5547660	·7618788	·9809189	1·2629330
	4	897680	·1849220	·1904697	·2452297	·3536212
3	12	·2693040	·7396880	·9523485	1·2261486	1·6165542
	4	897680	·1849220	·2380871	·3433216	·4546559
4	16	·3590720	·9246100	1·1904356	1·5694702	2·0712101
	4	897680	·2311525	·3333219	·4414135	·6062078
5	20	·4488400	1·1557625	1·5237575	2·0108837	2·6774179
	5	·1122100	·3236135	·4285568	·5885513	·8587944
6	25	·5610500	1·4793760	1·9523143	2·5994350	3·5362123
	7	·1570939	·4160745	·5714090	·8337810	1·2124157
7	32	·7181439	1·8954505	2·5237233	3·4332160	4·7486280
	9	·2019779	·5547661	·8094961	1·1771026	·17681062
8	41	·9201218	2·4502166	3·3332194	4·6103186	6·5167342
	12	·2693039	·7859186	1·1428181	1·7166080	·25763833
9	53	1·1894257	3·2361352	4·4760375	6·3269266	9·0931175
	17	·3815139	1·1095321	1·6666097	2·5013430	3·3846604
10	70	1·5709396	4·3456673	6·1426472	8·8282696	12·4777779
	24	·5386078	1·6180677	2·4284884	3·2860781	3·8393163
11	94	2·1095474	5·9637350	8·5711356	12·1143477	16·3170942
	35	·7854697	2·3577557	3·1903671	3·7274916	4·5465588
12	129	2·8950171	8·3214907	11·7615027	15·8418393	20·8636530
	51	1·1445416	3·0974438	3·6189239	4·4141348	5·8094918
13	180	4·0395587	11·4189345	15·3804266	20·2559741	26·6731448

$x < y$

$y - x$	d_x	$y = 104$	$y = 103$	$y = 102$	$y = 101$	$y = 100$
0	·5	·0224420	·1849220	·3809394	·5885513	·8082771
	·5	224420	0	0	0	0
1	1·0	·0448840	·1849220	·3809394	·5885513	·8082771
	·0	0	0	0	0	·4041386
2	1·0	·0448840	·1849220	·3809394	·5885513	1·2124157
	·0	0	0	0	·2942757	·4041386
3	1·0	·0448840	·1849220	·3809394	·8828270	1·6165543
	·0	0	0	·1904697	·2942757	·4041386
4	1·0	·0448840	·1849220	·5714091	1·1771027	2·0206929
	·0	0	924610	·1904697	·2942757	·8082771
5	1·0	·0448840	·2773830	·7618788	1·4713784	2·8289700
	·5	224420	924610	·1904697	·5885513	1·2124157
6	1·5	·0673260	·3698440	·9523485	2·0599297	4·0413857
	·5	224420	924610	·3809394	·8828270	1·6165543
7	2·0	·0897680	·4623050	1·3332879	2·9427567	5·6579400
	·5	224420	·1849220	·5714090	1·1771026	2·8289699
8	2·5	·1122100	·6472270	1·9046969	4·1198593	8·4869099
	1·0	448840	·2773830	·7618787	2·0599296	3·6372471
9	3·5	·1570940	·9246100	2·6665756	6·1797889	12·1241570
	1·5	673260	·3698440	1·3332877	2·6484809	2·8289699
10	5·0	·2244200	1·2944540	3·9998633	8·8282698	14·9531269
	2·0	897680	·6472271	1·7142271	2·0599296	·8082771
11	7·0	·3141880	1·9416811	5·7140904	10·8881994	15·7614040
	3·5	·1570940	·8321491	1·3332877	·5885513	4·8496627
12	10·5	·4712820	2·7738302	7·0473781	11·4767507	20·6110667
	4·5	·2019779	·6472271	·3809394	3·5313078	5·2538013
13	15·0	·6732599	3·4210573	7·4283175	15·0080585	25·8648680

$x \gamma y$

$x-y$	s_y	$x=99$	$x=98$	$x=97$	$x=96$	$x=95$
0	20 5	10406568 2601642	20097684 5627352	35329050 9936295	58279131 17057307	108635143 34845235
1	25 7	13008210 3642299	25725036 7235166	45265345 13248394	75336438 24164518	143480378 49193273
2	32 9	16650509 4682956	32960202 9646889	58513739 18768558	99500956 34114613	192673651 71740189
3	41 12	21333465 6243941	42607091 13666425	77282297 26496787	133615569 49750478	264413840 104535704
4	53 17	27577406 8845583	56273516 19293777	103779084 38641148	183366047 72493553	368949544 137331219
5	70 24	36422989 12487882	75567293 28136758	142420232 56305673	255859600 95236629	506280763 155778696
6	94 35	48910871 18211494	103704051 40999276	198725905 73970198	351096229 108029609	662059459 184474772
7	129 51	67122365 26536748	144703327 53861794	272696103 83906493	459125838 127929800	846534231 235777764
8	180 67	93659113 34862003	198565121 61096961	356602596 99362952	587055638 163465856	1082251995 276712158
9	247 76	128521116 39544958	259662082 72351664	455965548 126963772	750521494 191894700	1358964153 305408234
10	323 90	168066074 46829556	332013746 92449348	582929320 149044428	942416194 21794892	1664372387 32054589
11	413 115	214895630 59837766	424463094 108527496	731973748 164500887	1154211086 230273640	1996426976 364850104
12	528 135	274733396 70244534	532990590 119782199	896474635 178853313	1384484726 253016716	2361277080 401745059
13	663	344977730	652772789	1075327948	1637501442	2763022139

 $x \angle y$

$y-x$	d_x	$y=99$	$y=98$	$y=97$	$y=96$	$y=95$
0	10 5	10406568 5203284	20097684 6699228	35329050 8832262	58279131 23311652	108635143 46557919
1	15 5	15609852 5203284	26796912 6699228	44161312 17664525	81590783 34967479	155193062 62077225
2	20 5	20813136 5203284	33496140 13398456	61825837 26496787	116558262 46623305	217270287 108635144
3	25 10	26016420 10406568	46894596 20097684	88322624 35329050	163181567 81590784	325905431 139673756
4	35 15	36422988 15609852	66992280 26796913	123651674 61825837	244772351 104902436	465579187 108635144
5	50 20	52032840 20813136	93789193 46894597	185477511 79490361	349674787 81590784	574214331 31088612
6	70 35	72845976 36422988	140683790 60293053	264967872 61825837	431265571 23311652	605252943 186231675
7	105 45	109268964 46829556	200976843 46894597	326793709 17664525	454577223 139869915	791484618 201750981
8	150 35	156098520 36422988	247871440 13398456	344458234 105987148	594447138 151525741	993235599 108635144
9	185 10	192521508 10406568	261269896 80390738	450445382 114819411	745972879 81590784	1101870743 108635144
10	195 60	202928076 62439408	341660634 87089966	565264793 61825837	827563663 81590784	1210505887 93115837
11	255 65	265367484 67642692	428750600 46894597	627090630 61825837	909154447 69934957	1303621724 155193062
12	320 35	333010176 36422988	475645197 46894597	688916467 52993574	979089404 116558263	1458814786 124154450
13	355	369433164	522539794	741910041	1095647667	1582969236

$y-x=0$	$y-x=1$	$y-x=2$	$y-x=3$
$d_{104}s_{104}v^{105},$	$d_{103}s_{104}v^{105},$	$d_{102}s_{104}v^{105},$	$d_{101}s_{104}v^{105},$
$d_{103}s_{103}v^{104},$	$d_{102}s_{103}v^{104},$	$d_{101}s_{103}v^{104},$	$d_{100}s_{103}v^{104},$
$d_{102}s_{102}v^{103},$	$d_{101}s_{102}v^{103},$	$d_{100}s_{102}v^{103},$	$d_{99}s_{102}v^{103},$

and so on. Calling for a moment $s_{104}v^{105}$, $s_{103}v^{104}$, $s_{102}v^{103}$, &c., P', Q', R', &c. respectively, the foregoing terms may be written thus:—

P' $d_{104},$	P' $d_{103},$	P' $d_{102},$	P' $d_{101},$
Q' $d_{103},$	Q' $d_{102},$	Q' $d_{101},$	Q' $d_{100},$
R' $d_{102},$	R' $d_{101},$	R' $d_{100},$	R' $d_{99},$

and their differences, taken from the rows, will be

P' $\Delta d_{104},$	P' $\Delta d_{103},$	P' $\Delta d_{102},$
Q' $\Delta d_{103},$	Q' $\Delta d_{102},$	Q' $\Delta d_{101},$
R' $\Delta d_{102},$	R' $\Delta d_{101},$	R' $\Delta d_{100},$

The operation here, then, it thus appears, will be altogether analogous to that in the former case. The terms in the several rows, being multiples of the quantities P', Q', R', &c. respectively, will be formed by the continuous addition to an initial term of series of multiples of the same quantities, by the successive values of Δd_x , which also is a tabulated series.

The preceding example (pages 6 and 7) shows the practical application of the foregoing precepts, the rate of interest being 3 per cent.

The example consists of two portions, corresponding to the relative magnitudes of x and y , x being greater than y in the former, and less than y in the latter. In the first portion, x is constant in each column, and decreases by a unit in passing from column to column. On the other hand, y is equal to x in the first row, and decreases by a unit in descending from each row to the next. As a consequence, the relation between x and y in each row is that indicated by the values in the column headed $x-y$. Thus, in the first row, proceeding from left to right, we have the values of the function corresponding to $x=104$, $y=104$; $x=103$, $y=103$, and so on; in the second row we have the values corresponding to $x=104$, $y=103$; $x=103$, $y=102$; and so of the succeeding rows. The column headed s_y , which is supposed to be written on a separate piece of paper or card, and consequently to be moveable, contains the successive values of s_y , taken from the table on page 110, with their differences, *i. e.*, the values of Δs_y , on alter-

nate lines. In the present position of this slip the values upon it have, opposite to each, in the column headed 104, the multiple by that value of the quantity we have called P, namely, $d_{104}v^{105}$, it being understood that the initial value and the multiples corresponding to the values of Δs_y only have been independently formed, continuous addition of these giving the multiples corresponding to the values of s_y , the accuracy of which there is consequently the power of determining at any point. If the slip be now conceived to be moved up one step, and so applied to the column headed 103, the same thing will hold. Each value on the slip will have opposite to it in this column the multiple of the quantity called Q, namely, $d_{103}v^{104}$, corresponding to that value; of which multiples, as before, the initial value and those corresponding to the successive values of Δs_y only have been independently formed, the others being derived from these by continuous addition. It will thus be understood, that in passing from each column to the next the slip has to be moved up one step, and in the position thus attained it will indicate the multiples to be inserted in the adjoining column. A second representation of it is introduced, in connection with the column headed 99, to show the position it will have assumed when the first five columns have been passed over.

The foregoing description applies, with but little change, to the remaining portion of the example. In regard to variation in passing from column to column and from row to row, y here takes the place of x in the former portion; and generally it will be found that in corresponding values the values of x and y are merely interchanged. The moveable slip here contains the successive values of d_x and their differences, and the multiples in the several columns are those of the quantities P', Q', R', &c. Here, too, the slip is shown in two different positions.

It seems hardly necessary to remark, that each portion of the computation will consist of 105 columns and 105 rows, and that each column, in passing from left to right, and each row in passing from upper to lower, will contain one value fewer than the preceding; so that both the last column and the last row will contain each but a single value.

It remains to describe the method of forming the addends, which, as has been shown, consist of a series of multiples of a quantity, which is the same for each column, but different for different columns, the multipliers being in the one case ($x > y$),

the terms of the series Δs_y , and in the other ($x < y$), the terms of the series Δd_x .

Refer to the series Δs_y . We notice, first, that none of the values (with the exception of the last two) consist of more than three figures. The required multiples then can be easily formed by the aid of a small table of the multiples corresponding to the first nine natural numbers. Secondly, a good many of the values occur twice, and some of them oftener, so that the number of multiples to be formed will always be less than the number of values in the column to be constructed. And, thirdly, if the values be arranged once for all, *in the order of their magnitude*, with space between them for their differences, on a separate slip, similar to that already described, it is evident that then the multiples wanted may be very easily formed by the continuous addition of still smaller multiples.

Similar remarks apply to the other series, Δd_x , corresponding to the terms of which, also, multiples have to be formed. The formation in this case, however, is much easier. Only seven terms of the series consist of more than two figures; as many as twenty of them are zero; most of the others are frequently repeated, and of nearly the whole of these the constant difference is .5. There are, in fact, in no case more than twenty-four distinct multiples to be formed. It is to be noted, moreover, in reference to this series, that a good many of its terms are negative. The multiples corresponding to these terms must consequently be *subtracted* instead of *added*; and this change of affection should be indicated in setting them down by prefixing the negative sign, or by writing them in red ink.

To return now to the example. The values formed, it will be borne in mind, have reference each to a single year only, and are consequently not those that are requisite to be tabulated. The values for tabulation are those having reference to the *whole after lifetime at each age*. These values obviously consist of the successive sums formed by continuous addition of the values formed as above, in which x and y have a common difference. The formation of a few of these final values, for the first six differences of x and y , is shown on the opposite page.

This construction needs little explanation. The addends in the several columns will be found in the successive rows in the previous formation. The terms in the present formation, in which the values of x and y are interchanged, will be found occupying corresponding positions in the two portions; and addition of any two of these

$x \geq y$

x	$x-y=0$	$x-y=1$	$x-y=2$	$x-y=3$	$x-y=4$	$x-y=5$
104	·0224420 ·1849220	·0897680 ·3698440	·1795360 ·5547660	·2693040 ·7396880	·3590720 ·9246100	·4488400 1·1557625
103	·2073640 ·3809394	·4596120 ·5714091	·7343020 ·7618788	1·0089920 ·9523485	1·2836820 1·1904356	1·6046025 1·5237575
102	·5883034 ·5885513	1·0310211 ·7847351	1·4961808 ·9809189	1·9613405 1·2261486	2·4741176 1·5694702	3·1283600 2·0108837
101	1·1768547 ·8082771	1·8157562 1·0103464	2·4770997 1·2629330	3·1874891 1·6165542	4·0435878 2·0712101	5·1392437 2·6774179
100	1·9851318 1·0406568	2·8261026 1·3008210	3·7400327 1·6650509	4·8040433 2·1333465	6·1147979 2·7577406	7·8166616 3·6422989
99	3·0257886 2·0097684	4·1269236 2·5725036	5·4051836 3·2960202	6·9373898 4·2607091	8·8725385 5·6273516	11·4589605 7·5567293
98	5·0355570 3·5329050	6·6994272 4·5265345	8·7011038 5·8513739	11·1980989 7·7282297	14·4998901 10·3779084	19·0156898 14·2420232
97	8·5684620 5·8279131	11·2259617 7·5336438	14·5524777 9·9500956	18·9263286 13·3615569	24·8777985 18·3366047	33·2577130 25·5859600
96	14·3963751 10·8635143	18·7596055 14·3480378	24·5025733 19·2673651	32·2878855 26·4413840	43·2144032 36·8949544	58·8436730 50·6280763
95	25·2598894	33·1076433	43·7699384	58·7292695	80·1093576	109·4717493

 $x < y$

y	$y-x=0$	$y-x=1$	$y-x=2$	$y-x=3$	$y-x=4$	$y-x=5$
104	·0224420 ·1849220	·0448840 ·1849220	·0448840 ·1849220	·0448840 ·1849220	·0448840 ·1849220	·0448840 ·2773830
103	·2073640 ·3809394	·2298060 ·3809394	·2298060 ·3809394	·2298060 ·3809394	·2298060 ·5714091	·3222670 ·7618788
102	·5883034 ·5885513	·6107454 ·5885513	·6107454 ·5885513	·6107454 ·8828270	·8012151 1·1771027	1·0841458 1·4713784
101	1·1768547 ·8082771	1·1992967 ·8082771	1·1992967 1·2124157	1·4935724 1·6165543	1·9783178 2·0206929	2·5555242 2·8289700
100	1·9851318 1·0406568	2·0075738 1·5609852	2·4117124 2·0813136	3·1101267 2·6016420	3·9990107 3·6422988	5·3844942 5·2032840
99	3·0257886 2·0097684	3·5685590 2·6796912	4·4930260 3·3496140	5·7117687 4·6894596	7·6413095 6·6992280	10·5877782 9·3789193
98	5·0355570 3·5329050	6·2482502 4·4161312	7·8426400 6·1825837	10·4012283 8·8322624	14·3405375 12·3651674	19·9666975 18·5477511
97	8·5684620 5·8279131	10·6643814 8·1590783	14·0252237 11·6558262	19·2334907 16·3181567	26·7057049 24·4772351	38·5144486 34·9674787
96	14·3963751 10·8635143	18·8234597 15·5193062	25·6810499 21·7270267	35·5516474 32·5905431	51·1829400 46·5579187	73·4819273 57·4214331
95	25·2598894	34·3427659	47·4080786	68·1421905	97·7408587	130·9033604

If in the foregoing construction v be made equal to unity (which leads to $r=0$, *i. e.*, to the hypothesis of money bearing no interest), the construction will be much facilitated. The results of columns E and F will then indicate probabilities of survivorship, while column M will become identical with column D, since the number of failures *after* a specified age is necessarily equal to the number alive *at* the same age.

II.—Construction in Logarithms.

Recurring to the symbolical expressions on page 110 for the leading terms of several adjoining columns belonging to the case $x > y$, namely,

$x-y=0$	$x-y=1$	$x-y=2$	$x-y=3$
$d_{104}s_{104}v^{105},$	$d_{104}s_{103}v^{105},$	$d_{104}s_{102}v^{105},$	$d_{104}s_{101}v^{105},$
$d_{103}s_{103}v^{104},$	$d_{103}s_{102}v^{104},$	$d_{103}s_{101}v^{104},$	$d_{103}s_{100}v^{104},$
$d_{102}s_{102}v^{103},$	$d_{102}s_{101}v^{103},$	$d_{102}s_{100}v^{103},$	$d_{102}s_{99}v^{103},$

we have to examine now, not the differences, but the ratios of those terms. Take a column—the first. Here we have

$$\frac{d_{103}s_{103}v^{104}}{d_{104}s_{104}v^{105}} = \frac{d_{103}}{d_{104}} \cdot \frac{s_{103}}{s_{104}} (1+r); \quad \frac{d_{102}s_{102}v^{103}}{d_{103}s_{103}v^{104}} = \frac{d_{102}}{d_{103}} \cdot \frac{s_{102}}{s_{103}} (1+r);$$

and we should find similar expressions for the ratios in the other columns. It will thus appear that the addends by the use of which, setting out with the logarithms of the initial terms, we should form continuously the logarithms of the successive terms in each column, are

$$\begin{aligned} (x-y=0), & \Delta \log. d_{104} + \Delta \log. s_{104} + \log. (1+r), \Delta \log. d_{103} + \Delta \log. s_{103} + \log. (1+r), \&c. \\ (x-y=1), & \Delta \log. d_{104} + \Delta \log. s_{103} + \log. (1+r), \Delta \log. d_{103} + \Delta \log. s_{102} + \log. (1+r), \&c. \\ (x-y=2), & \Delta \log. d_{104} + \Delta \log. s_{102} + \log. (1+r), \Delta \log. d_{103} + \Delta \log. s_{101} + \log. (1+r), \&c. \end{aligned}$$

and so on. There is nothing formidable here. Additions of three lines will suffice, since the constant, $\log. (1+r)$, can be combined once for all with the terms of one or other of the series $\Delta \log. d_x$ or $\Delta \log. s_y$. But I use this method only for verification. There is an easier method for the principal formation.

Examine the ratios in the several rows. From the first row we have—

$$\frac{d_{104}s_{103}v^{105}}{d_{104}s_{104}v^{105}} = \frac{s_{103}}{s_{104}}; \quad \frac{d_{104}s_{102}v^{105}}{d_{104}s_{103}v^{105}} = \frac{s_{102}}{s_{103}}; \quad \frac{d_{104}s_{101}v^{105}}{d_{104}s_{102}v^{105}} = \frac{s_{101}}{s_{102}}, \&c.;$$

from the second—

$$\frac{d_{103}s_{102}v^{104}}{d_{103}s_{103}v^{104}} = \frac{s_{102}}{s_{103}}; \quad \frac{d_{103}s_{101}v^{104}}{d_{103}s_{102}v^{104}} = \frac{s_{101}}{s_{102}}; \quad \frac{d_{103}s_{100}v^{104}}{d_{103}s_{101}v^{104}} = \frac{s_{100}}{s_{101}}, \text{ \&c.};$$

and so on. Hence, taking the logarithms of these ratios, the addends for the continuous formation of the logarithms of the values occupying the several rows will be the terms of the series $\Delta \log. s_y$.

A like procedure to that employed above will show, that in the case $x < y$ the addends for the formation in columns are series of terms of the same form as those obtained for the case $x > y$, the only difference being that the terms of the series $\Delta \log. d_x$ and $\Delta \log. s_y$ are interchanged; and that for the formation in rows the addends are the terms of the series $\Delta \log. d_x$.

The following table contains the portions of the several auxiliary series here requisite, which come into operation in the examples to be presently given.

$x \ \& \ y$	Log. s_y	$\Delta \log. s_y$	$\nabla \log. s_y$	Log. d_x	$\Delta \log. d_x$	$\nabla \log. d_x$
104	0.0000000	6020600	6148972	1.6989700	3010300	3138672
103	.6020600	3010300	3138672	0.0000000	0000000	0128372
102	.9030900	1760912	1889285	.0000000	0000000	0128373
101	1.0791812	1249388	1377760	.0000000	0000000	0128372
100	.2041200	0969100	1097472	.0000000	0000000	0128372
99	.3010300	0969100	1097472	.0000000	1760913	1889285
98	.3979400	1072100	1200473	.1760913	1249387	1377760
97	.5051500	1076339	1204711	.3010300	0969100	1097472
96	.6137839	1114920	1243292	.3979400	1461280	1589652
95	.7242759	1208221	1336593	.5440680	1549020	1677392
94	.8450980	1280299	1408672	.6989700	1461280	1589653
93	.9731279	1374618	1502990	.8450980	1760913	1889285
92	2.1105897	1446828	1575200	1.0211893	1549020	1677392
91	.2552725	1374245	1502617	.1760913	0910804	1039176
90	.3926970	1165055	1293428	.2671717	0228629	0357002
89	.5092025	1067476	1195848	.2900346	1165056	1293428
88	.6159501	1066838	1195210	.4065402	0986098	1114470
87	.7226339	0988796	1117168	.5051500	0450784	0579156
86	.8215135	0880425	1008798	.5502284	0408362	0536735
85	.9095560	0790030	0918402	.5910646	0321847	0450219
84	.9885590			.6232493		

The formation of this table is sufficiently obvious,* with the exception of the columns headed $\nabla \log. s_y$ and $\nabla \log. d_x$. The values in these columns are the corresponding values in the immediately preceding columns respectively, increased by $\log. (1+r) = .0128372$. The succeeding figures of this constant logarithm being 247, or say 25, the requisite correction is made by

* $\Delta \log. s_y$ and $\Delta \log. d_x$ must not be confounded with $\log. \Delta s_y$ and $\log. \Delta d_x$. The difference of the logarithm of a number is by no means the same thing as the logarithm of the difference of that number.

increasing the third and every fourth following term of the series in which this logarithm enters by a unit in the last place. But for the necessity for this correction, it would suffice to combine the constant logarithm with the terms of only one of the series $\Delta \log. s_y$, $\Delta \log. d_x$. The corrections would in such case fall in the proper places in the values formed for verification, only when the differences of age are of the form $4n$.

The initial and verification values are formed as follows :—

Initial and Verification Values.

$x \nabla y$				$x \angle y$			
$x - y = 0$		$x - y = 13$		$y - x = 0$		$y - x = 13$	
Log. v^{105}	6520914	Log. v^{105}	6520914	Log. v^{105}	6520914	Log. v^{105}	6520914
„ d_{104}	6989700	„ d_{104}	6989700	„ s_{104}	0000000	„ s_{104}	0000000
„ s_{104}	0000000	„ s_{91}	2552725	„ d_{104}	6989700	„ d_{91}	1760913
104.104	3510614	104.91	6063339	104.104	3510614	91.104	8281827
$\nabla \log. d_{104}$	3138672	$\nabla \log. d_{104}$	3138672	$\nabla \log. s_{104}$	6148972	$\nabla \log. s_{104}$	6148972
Δ „ s_{104}	6020600	Δ „ s_{91}	1374245	Δ „ d_{104}	3010300	Δ „ d_{91}	0910804
103.103	2669886	103.90	0576256	103.103	2669886	90.103	5341603
$\nabla \log. d_{103}$	0128372	$\nabla \log. d_{103}$	0128372	$\nabla \log. s_{103}$	3138672	$\nabla \log. s_{103}$	3138672
Δ „ s_{103}	3010300	Δ „ s_{90}	1165055	Δ „ d_{103}	0000000	Δ „ d_{90}	0228629
102.102	5808558	102.89	1869683	102.102	5808558	89.102	8708904
$\nabla \log. d_{102}$	0128373		0128373		1889285		1889285
Δ „ s_{102}	1760912		1067476		0000000		1165056
101.101	7697843	101.88	3065532	101.101	7697843	88.101	1763245
$\nabla \log. d_{101}$	0128372		0128372		1377760		1377760
Δ „ s_{101}	1249388		1066838		0000000		0986098
100.100	9075603	100.87	4260742	100.100	9075603	87.100	4127103
$\nabla \log. d_{100}$	0128372		0128372		1097472		1097472
Δ „ s_{100}	0969100		0988796		0000000		0450784
99.99	0173075	99.86	5377910	99.99	0173075	86.99	5675359
<i>Verification of the above Values.</i>							
Log. v^{100}	7162775	Log. v^{100}	7162775	Log. v^{100}	7162775	Log. v^{100}	7162775
„ d_{99}	0000000	„ d_{99}	0000000	„ s_{99}	3010300	„ s_{99}	3010300
„ s_{99}	3010300	„ s_{86}	8215135	„ d_{99}	0000000	„ d_{86}	5502284
	0173075		5377910		0173075		5675359

The initial values are, of course, the same in both compartments. I have, nevertheless, formed them separately for the sake of symmetry, and to show more distinctly how the addend series vary in passing from column to column.

I now introduce an example of the actual formation :—

$x \geq y$

$x - y$	$x = 104$	$x = 103$	$x = 102$	$x = 101$	$x = 100$	$x = 99$
0	3510614 6020600	2669886 3010300	5808558 1760912	7697843 1249388	9075603 0969100	0173075 0969100
1	9531214 3010300	5680186 1760912	7569470 1249388	8947231 0969100	0044703 0969100	1142175 1072100
2	2541514 1760912	7441098 1249388	8818858 0969100	9916331 0969100	1013803 1072100	2214275 1076339
3	4302426 1249388	8690486 0969100	9787958 0969100	0885431 1072100	2085903 1076339	3290614 1114920
4	5551814 0969100	9659586 0969100	0757058 1072100	1957531 1076339	3162242 1114920	4405534 1208221
5	6520914 0969100	0628686 1072100	1829158 1076339	3033870 1114920	4277162 1208221	5613755 1280299
6	7490014 1072100	1700786 1076339	2905497 1114920	4148790 1208221	5485383 1280299	6894054 1374618
7	8562114 1076339	2777125 1114920	4020417 1208221	5357011 1280299	6765632 1374618	8268672 1446828
8	9638453 1114920	3892045 1208221	5228638 1280299	6637310 1374618	8140300 1446828	9715500 1374245
9	0753373 1208221	5100266 1280299	6508937 1374618	8011928 1446828	9587128 1374245	1089745 1165055
10	1961594 1280299	6380565 1374618	7883555 1446828	9458756 1374245	0961373 1165055	2254800 1067476
11	3241893 1374618	7755183 1446828	9330383 1374245	0833001 1165055	2126428 1067476	3322276 1066838
12	4616511 1446828	9202011 1374245	0704628 1165055	1998056 1067476	3193904 1066838	4389114 0988796
13	6063339	0576256	1869683	3065532	4260742	5377910

$x < y$

$y - x$	$y = 104$	$y = 103$	$y = 102$	$y = 101$	$y = 100$	$y = 99$
0	3510614 3010300	2669886 0	5808558 0	7697843 0	9075603 0	0173075 1760913
1	6520914 0	2669886 0	5808558 0	7697843 0	9075603 1760913	1933988 1249387
2	6520914 0	2669886 0	5808558 0	7697843 1760913	0836516 1249387	3183375 0969100
3	6520914 0	2669886 0	5808558 1760913	9458756 1249387	2085903 0969100	4152475 1461280
4	6520914 0	2669886 1760913	7569471 1249387	0708143 0969100	3055003 1461280	5613755 1549020
5	6520914 1760913	4430799 1249387	8818858 0969100	1677243 1461280	4516283 1549020	7162775 1461280
6	8281827 1249387	5680186 0969100	9787958 1461280	3138523 1549020	6065303 1461280	8624055 1760913
7	9531214 0969100	6649286 1461280	1249238 1549020	4687543 1461280	7526583 1760913	0384968 1549020
8	0500314 1461280	8110566 1549020	2798258 1461280	6148823 1760913	9287496 1549020	1933988 0910804
9	1961594 1549020	9659586 1461280	4259538 1760913	7909736 1549020	0836516 0910804	2844792 0228629
10	3510614 1461280	1120866 1760913	6020451 1549020	9458756 0910804	1747320 0228629	3073421 1165056
11	4971894 1760913	2881779 1549020	7569471 0910804	0369560 0228629	1975949 1165056	4238477 0986098
12	6732807 1549020	4430799 0910804	8480275 0228629	0598189 1165056	3141005 0986098	5224575 0450784
13	8281827	5341603	8708904	1763245	4127103	5675359

After what has been said, this example seems to require but little explanation. The arrangement of the values is precisely the same as in the preceding formation in numbers. In the first compartment x is constant in each column, and in the second y is constant, while in both the differences of age are constant in the several rows. The values here formed are consequently the logarithms of the values occupying corresponding positions in the preceding formation. In practice it will be proper to leave the alternate columns unoccupied, to receive the natural numbers, which it is unnecessary here to take out, since, as just mentioned, we already have them in the first formation.

Refer to either portion of the present example—say the first. The addends in the several columns are the successive terms of the same series ($\Delta \log. s_y$), commencing in each column with the term immediately following that with which a commencement was made in the preceding column. Each term consequently may be traced, from its occurrence in the first column, in a diagonally ascending line, till it appears for the last time in the first row. It is comparatively easy then, when the addends have been properly inserted in the first column, to insert them throughout, since a single reference for each value will suffice for every occurrence of that value on the same opening of the paper made use of. But the labour of writing the addends may be spared by writing them once for all upon one or more perforated cards of the form here shown.

The manner of using this card will be obvious from comparison of it with the first portion of the example just given. For the first column it will be placed with the first opening over the initial value in that column. Continuous addition then will, it is apparent, produce precisely the same values as already appear in the column in question. To form the second column, the card will be placed so that the initial value in that column shall appear through the second opening, opposite 103, and continuous addition will, as before, produce the required values. To make this more clear,

	$\Delta \log. s_y$
104	
	6020600
103	
	3010300
102	
	1760912
101	
	1249388
100	
	0969100
99	
	0969100
98	
	1072100
97	
	1076339

the card is here shown in three different positions, adapted to the formation of the leading terms of the first three columns.

$\Delta \log. s_y$		$\Delta \log. s_y$		$\Delta \log. s_y$	
104	3510614	104		104	
	6020600		6020600		6020600
103	9531214	103	2669886	103	
	3010300		3010300		3010300
102	2541514	102	5680186	102	5808558
	1760912		1760912		1760912
101	4302426	101	7441098	101	7569470
	1249388		1249388		1249388
100	5551814	100	8690486	100	8818858
	0969100		0969100		0969100
99	6520914	99	9659586	99	9787958
	0969100		0969100		0969100
98		98	0628686	98	0757058
	1072100		1072100		1072100
97		97		97	1829158
	1076339		1076339		1076339

In a previous part of this paper I intimated my preference for the first of the two methods of construction that have now been described. The reason of this preference may be now briefly stated. It is simply this—that when the method in question is

employed we have the benefit of a continuous process, and consequently the power of thorough verification, up to a later stage than when the logarithmic method is made use of. By the first method we form continuously the *actual values* of the functions, of which, by the other method, we form only *the logarithms*; and as the operation of taking out the numbers corresponding to these logarithms is one to which an *infallible* check cannot be applied, it seems to follow that we have the greatest security against error by employing the method of construction in numbers. It is true that this method is attended with *a little* more labour than the other; but, were the additional labour very much more than it is, I should consider it amply compensated by the advantage in regard to security against error which, as we have just seen, this method possesses.

Another distinction between the two methods deserves to be mentioned. When the logarithmic method is employed we can obtain no more than seven figures in the several results. Seven figures may or may not be sufficient. On this point different computers will entertain different views. Those who desire a greater number, however, or to ensure greater accuracy with the same number, can have their desire gratified to the fullest extent by employing the method of construction in numbers. The number of figures that may be obtained true by this method is, for all practicable purposes, unlimited.

And now, before I close, I would direct attention to a mechanical aid, the use of which imparts an almost magical facility to the most complex and extensive computations, and in the absence of which even the best devised methods of applying formulæ at all complex, or requiring the employment of many figures, become exceedingly perplexing and irksome. I refer to the employment of paper, ruled in squares, each square being adapted to receive a single figure. By this means not only does each figure necessarily fall in its proper place, and so facilitate subsequent proceedings, but values that occur *periodically*, and have to be *worked up to*, as it were, in the course of continuous or other operations, can be inserted in their places in the outset, so as to avoid all necessity for interruption in the work. But one must try this paper to be able fully to appreciate its great advantages, and the facilities it affords. I believe I am warranted in saying, that by the employment of it in such operations as have been described in the present paper, the labour that would otherwise be necessary will be found to be reduced one half, while accuracy will be most materially

promoted. The paper may be obtained by order of any stationer. That which I use (foolscap) has about 70 horizontal lines and 46 vertical lines on a page.

I am quite aware that this paper is known to and used by a good many actuaries, as well as in astronomical computations; but I have recently had occasion to become acquainted with so many instances in which it is not known, that I believe I shall be doing good service in thus prominently directing attention to its advantages.

I have now finished. I am quite prepared to learn that the practised computer, whatever opinion he may entertain of the methods I have endeavoured to develope, will think that I have been unnecessarily minute in my details as to their application. I admit it. *For him* I have gone into unnecessary detail. But I have had chiefly in view the younger members of the Institute, whom I would fain stimulate to enter upon the exercise of the construction of tables, and I have been anxious to remove everything that might prove an obstruction in their way. If two, three, or four of our younger members would unite for the construction of a complete set of tables of the kind now described, the labour would be comparatively light; and, besides the benefit they would thus confer on the Profession generally, I feel very sure that, at the conclusion of their task, they would be ready to acknowledge that the exercise had been the most beneficial to themselves of any they ever entered upon. They would find they had acquired such an intimate acquaintance with the structure and properties of the tables, that they could apply them to practical purposes with a facility and confidence which without this preparation long experience only could have imparted. I should be glad to afford, to any of our friends disposed to act on this recommendation, all the assistance in my power in the way of information or suggestion.

In conclusion, I think it necessary to mention that the methods developed in the foregoing paper are identical in principle with methods I have elsewhere described for like purposes.
